SCHOOL OF MATHEMATICAL SCIENCES, SHANGHAI JIAO TONG UNIVERSITY

Applied Functional Analysis: Homework 3*

Mounir HAJLI

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Exercise 1

Let (X, d) be a metric space.

1. Let (x_k) be a sequence of points in X converging to a limit $a \in X$. Prove that $K \cup \{x_k | k \in \mathbb{N}\}$ is compact.

2. Let Y be a metric space and $f : X \to Y$ an application. We suppose that the restriction of f to any compact subset of X is continuous. Prove that f is continuous.

Exercise 2

Let X consist of two points a and b, put $\mu(\{a\}) = \mu(\{b\}) = \frac{1}{2}$

Exercise 3

Let E be a normed space, A, B subsets of E. We set $A + B = \{x + y | x \in A, y \in B\}$

1. Suppose$A$and B are compacts. Show that $A + B$ is compact.

2. Suppose that A is compact and Fclosed. Prove that $A + B$ is closed in E.

*Due 11 October

Exercise 4

Let K be a compact non empty of metric space (X,d) and U an open subset of X containing K. Show that there exists $r>0$ such that for any $x \in X$, we have

$$d(x,K) < r \implies x \in U.$$

(Hint): consider the application $x \to d(x, X \setminus U)$ defined on K

Exercise 5

Let $p, q > 1$ with

$$\frac{1}{p} + \frac{1}{q} = 1.$$

1. a) Show the following : For any $a, b > 0$, we have

$$a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q} \tag{0.1}$$

$$(\frac{a+b}{2})^p \leq \frac{1}{2}(a^p + b^p). \tag{0.2}$$

2. Consider the space $l^p(\mathbb{N})$ (See the notes).
   a) Show that $l^p(\mathbb{N})$ is a vector space.
   b) Prove that

$$\sum_{n=1}^{\infty} |x_n||y_n| \leq \|x\|_p \|y\|_q \quad \forall x = (x_n) \in l^p(\mathbb{N})\ \forall y = (y_n) \in l^q(\mathbb{N}).$$

   c) Prove that the following map

$$l^p(\mathbb{N}) \to l^1(\mathbb{N}), (x_n)_n \mapsto (x_n y_n)_n$$

   where $(y_n)_n \in l^q(\mathbb{N})$, is well defined and continuous.

3. Let $x, y \in l^p(\mathbb{N})$, Prove that

$$(\sum_{n\in\mathbb{N}} |x_n + y_n|^p)^{\frac{1}{p}} \leq (\sum_{n\in\mathbb{N}} |x_n|^p)^{\frac{1}{p}} + (\sum_{n} |y_n|^p)^{\frac{1}{p}}$$

4. Show that $\|\cdot\|_p$ is a norm on $l^p(\mathbb{N})$.

5. Prove that $(l^p(\mathbb{N}), \|\cdot\|_p)$ is Banach. Hint: Take a Cauchy sequence $(x^{(k)})_{k\in\mathbb{N}}$, and show that the coordinates form a Cauchy sequence.

   We have $|x_n^{(k)} - x_n^{(l)}| \leq \|x^{(k)} - x^{(l)}\|_p$ for any $n, k, l \in \mathbb{N}$, why? Then for any n , $(x_n^{(k)})_k$ converges to a limit z_n. Put $z = (z_n)_{n\in\mathbb{N}}$.

$$\|x^{(l)} - z\|_p \leq \|x^{(l)} - x^{(k)}\|_p + \|x^{(k)} - z\|_p \tag{0.3}$$

Now, let $\epsilon > 0, \exists N$ so that $\|x^{(l)} - x^{(k)}\|_p < \epsilon$ for $k, l \geq N$. Fix $k \geq N$, then there exists $M > 0$ such that

$$\sum_{n=M}^{\infty} |x_n^{(k)} - z_n|^p < \epsilon^p.(Why?)$$

This give $\sum_{n=N}^{\infty} |x_n^{(l)} - z_n|^p \leq 2\epsilon, \forall l \geq N$ (use the triangle inequality). Consider the sum $\sum_{n=0}^{N-1} |x_n^{(l)} - z_n|^p$, we can find $l \gg 1$ such that this finite sum is $\leq \epsilon$. and then $\|x^{(l)} - z\|_p \leq 2^{\frac{1}{p}}\epsilon$. Use (0.3) to conclude.

6. We set $l_0(\mathbb{N}) = \{x = (x_n)_{n\in\mathbb{N}} | \exists N \in \mathbb{N}, x_n = 0 \forall n \geq N\}$. Show that $l_0(\mathbb{N})$ is a vector space of which is dense in $l^p(\mathbb{N})$.

Exercise 6

Let E be a Banach space and F a closed subspace. For each coset $x + F$ of F, define $|x + F| = \inf|x + y|$ for $y \in F$. Show this defines a norm on the quotient space E/F, and the natural map $E \to E/F$ is continuous.